A Newton-Kantorovich-type Theorem in Quasi-Metric Spaces

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Problem Setup

Let $(\mathbb{M}, \mathrm{d})$ be a (quasi-)metric space

Consider a function $F : \mathbb{M} \to \mathbb{R}^n$

Want to find $\bar{x}$ such that $F(\bar{x}) = 0$

1. Superlinear Convergence Result
2. Existence (Inverse Function) Theorem

(Quasi-)Metric Spaces

Quasi-Metric Spaces

Definition

QMS1:

$$\forall x, y \in \mathbb{M}, x \neq y, \quad \mathrm{d}(x, y) > 0$$

QMS2:

$$\forall x \in \mathbb{M}, \quad \mathrm{d}(x, x) = 0$$

QMS3:

$$\forall x, y, z \in \mathbb{M}, \quad \mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$$

Definition

MS4:

$$\forall x, y \in \mathbb{M}, \quad \mathrm{d}(x, y) = \mathrm{d}(y, x)$$

Properties of Quasi-Metric Spaces

1. d induces a topology
2. convergence, convergence rates
3. completeness, Cauchy sequences
4. Banach Fixed Point Theorem

Banach Fixed Point Theorem

Definition

A mapping $T : \mathbb{M} \rightrightarrows \mathbb{M}$ is called a quasi-contraction at $\bar{x}$ if $\exists c < 1$

$$\forall x \in \mathbb{M}, \forall y \in T(x), \quad \mathrm{d}(y, \bar{x}) \leq c \, \mathrm{d}(x, \bar{x})$$

Theorem

Banach Fixed Point Let $\mathbb{M}$ *quasi-metric,* $T : V \subseteq \mathbb{M} \rightrightarrows \mathbb{M}$ *quasi-contraction at* $\bar{x} \in \mathbb{M}$ *with* $T(\bar{x}) = \bar{x}$*, any sequence* $\{x^k\}$*,* $x^{k+1} \in T(x^k)$ *converges linearly to the unique fixed point* $\bar{x}$

Algebraic Constructs

Algebraic constructs

The constructions are based on Euclidean/Riemannian spaces

Definition

$H : \mathbb{M} \times \mathbb{M} \to \mathbb{R}^n$ is pseudo-linear $\Rightarrow \forall x \in \mathbb{M}, H(x, x) = 0$

The set of pseudo-linear maps: $S_n(\mathbb{M})$

Definition

$H : \mathbb{M} \to \mathbb{M} \to \mathbb{R}^n$ pseudo-linear is inversely compatible if $\exists H^- : \mathbb{M} \times \mathbb{R}^n \to \mathbb{M}$ and $|||H^-||| > 0 \Rightarrow$

$$H^-(x, 0) = x$$
$$\mathrm{d}(H^-(x, v), H^-(y, w)) \leq |||H^-||| \|v - w - H(x, y)\|$$

The set of inversely compatible maps: $GS_n(M)$

Motivation

In $\mathbb{R}^n$ two points determine a vector, $(x, y) \mapsto y - x$

In $\mathcal{M}$ (a Riemannian manifold) two points determine a tangent vector, $(x, y) \mapsto \log_x y$

Euclidean	Riemannian	(Quasi-)Metric
vector	tangent vector	two points
linear map $T \in \mathbb{R}^{n \times n}$ $(x, y) \mapsto T(y - x)$	linear map $T \in \mathrm{Home}(T_x\mathcal{M})$ $(x, y) \mapsto T \log_x y$	pseudo-linear map $H \in S_n(\mathbb{M})$ $(x, y) \mapsto H(x, y)$
inverse map $(x, v) \mapsto x + T^{-}v$	inverse map $(x, v) \mapsto \exp_x T^{-1}v$	inversely compatible map $(x, v) \mapsto H^{-}(x, v)$

Example

Consider $\mathbb{R}^n$ and $T \in \mathbb{R}^{n\times n}$

We can consider T as a pseudo linear map

$$H(x,y) := T(y-x)$$

Further if T is invertible, it is inversely compatible

$$\begin{aligned}\mathrm{d}(H^-(x,v), H^-(y,w)) &= \|H^-(y,w) - H^-(x,v)\| \\ &= \|y + T^{-1}w - x - T^{-1}v\| \\ &\leq \|T^{-1}\| \|T(y-x) + w - v\| \\ &= \|T^{-1}\| \|v - w - T(y-x)\|.\end{aligned}$$

Further

$$\|H^-\| = \|T^{-1}\|$$

Newton Differentiability

Newton Differentiability in $\mathbb{R}^n$

Definition

$F : \mathbb{R}^n \to \mathbb{R}^n$ is *(weakly) Newton differentiable at $\bar{x}$* if there are $c \in \mathbb{R}(c \neq 0)$ and $\mathcal{H}F : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n \times n}$ with

$$\lim_{x \to \bar{x}} \sup_{H \in \mathcal{H}F(x)} \frac{\|F(x) - F(\bar{x}) - H(x - \bar{x})\|}{\|x - \bar{x}\|} = c$$

Definition

$F : \mathbb{R}^m \to \mathbb{R}^n$ is *uniformly (weakly) Newton differentiable at $\bar{x}$* if there are $c \in \mathbb{R}(c \neq 0)$, $\mathcal{H}F : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n \times n}$ such that $\forall \varepsilon > 0\ \exists \delta$, $\forall x \in \mathbb{R}^m$, $\forall y \in V$ with $\|x - y\| \leq \delta$,

$$\sup_{H \in \mathcal{H}F(x)} \frac{\|F(x) - F(\bar{x}) - H(x - \bar{x})\|}{\|x - \bar{x}\|} \leq c + \varepsilon$$

Examples

Proposition

$F : \mathbb{R}^n \to \mathbb{R}^n$, $F \in \mathcal{C}^1(\mathbb{R}^n)$, then F is Newton differentiable at any $x \in \mathbb{R}^n$ with a Newton differential $\mathcal{H}(x) := \{\nabla F(x)^T\}$ Further if ∇F is uniformly continuous on K, then F is uniformly Newton differentiable on K

Proposition

$f : \mathbb{R}^n \to \mathbb{R}$ μ-strongly convex $\mathcal{C}^1$ and L-smooth, then ∇f is weakly Newton differentiable at any x with a Newton differential $\mathcal{H}(\nabla f)(x) := \{\alpha \, \mathrm{I} \mid \frac{\sqrt{L^2+\alpha^2-2\alpha m}}{\alpha} \leq 1\}$

Examples

Definition

$F : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz *(uniformly semi-smooth on V)* *semi-smooth at x* if $\forall \varepsilon > 0$ ($\exists \delta$ such that $\forall x$, $\forall y \in V$ with $\|x - y\| \leq \delta$),

$$\frac{\|F'(x; x-y) - F'(y; x-y)\|}{\|x-y\|} \leq \varepsilon$$

Proposition

$F : \mathbb{R}^n \to \mathbb{R}^n$, F (uniformly semi-smooth on V) semi-smooth at x, then F is (uniformly Newton differentiable on V) Newton differentiable at x with a Newton differential $\mathcal{H}F(x) := \overline{\mathrm{conv}} \left\{ H \in \mathbb{R}^{n \times m} \mid \exists \{x^k\}_{k \in \mathbb{N}}, \lim_{k \to \mathbb{N}} \nabla F(x^k)^T = H \right\}$

Examples

Example[1]

$$F(x,y) = \begin{bmatrix} 4(x^2 - y) + 2(x-1) \\ -2(x^2 - y) \end{bmatrix} \quad \mathcal{H}F(x,y) = \left\{ 2 \begin{bmatrix} 1 + 4x^2 & -2x \\ -2x & 1 \end{bmatrix} \right\}$$

F is Newton differentiable at 0

Example

$$F(x) = \begin{cases} x & x \in \mathbb{Q}, \\ -x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \qquad \mathcal{H}F(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ -1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

F is Newton differentiable at 0

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[1] R. Bergmann et al. "The difference of convex algorithm on Hadamard manifolds"

Newton Differentiability in $\mathbb{R}^n$

Definition

$F : \mathbb{R}^n \to \mathbb{R}^n$ is *(weakly) Newton differentiable at $\bar{x}$* if there are $c \in \mathbb{R}(c \neq 0)$ and $\mathcal{H}F : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n\times n}$ with

$$\lim_{x\to\bar{x}} \sup_{H\in\mathcal{H}F(x)} \frac{\|F(x) - F(\bar{x}) - H(x-\bar{x})\|}{\|x-\bar{x}\|} = c$$

Definition

$F : \mathbb{R}^m \to \mathbb{R}^n$ is *uniformly (weakly) Newton differentiable at $\bar{x}$* if there are $c \in \mathbb{R}(c \neq 0)$, $\mathcal{H}F : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n\times n}$ such that $\forall \varepsilon > 0\ \exists \delta$, $\forall x \in \mathbb{R}^m$, $\forall y \in V$ with $\|x - y\| \leq \delta$,

$$\sup_{H\in\mathcal{H}F(x)} \frac{\|F(x) - F(\bar{x}) - H(x-\bar{x})\|}{\|x-\bar{x}\|} \leq c + \varepsilon$$

Newton Differentiability

Definition

$F : \mathbb{M} \to \mathbb{R}^n$ is *(weakly) Newton differentiable at $\bar{x}$* if there are $c \in \mathbb{R}(c \neq 0)$ and $\mathcal{H}F : \mathbb{R}^n \rightrightarrows S_n(\mathbb{M})$ with

$$\lim_{x \to \bar{x}} \sup_{H \in \mathcal{H}F(x)} \frac{\|F(x) - F(\bar{x}) - H(\bar{x}, x)\|}{\mathrm{d}(\bar{x}, x)} = c$$

Definition

$F : \mathbb{M} \to \mathbb{R}^n$ is *uniformly (weakly) Newton differentiable at $\bar{x}$* if there are $c \in \mathbb{R}(c \neq 0)$, $\mathcal{H}F : \mathbb{R}^n \rightrightarrows S_n(M)$ such that $\forall \varepsilon > 0\ \exists \delta$, $\forall x \in \mathbb{M}$, $\forall y \in V$ with $\mathrm{d}(x, y) \leq \delta$,

$$\sup_{H \in \mathcal{H}F(x)} \frac{\|F(x) - F(\bar{x}) - H(\bar{x}, x)\|}{\mathrm{d}(\bar{x}, x)} \leq c + \varepsilon$$

Calculus of Newton Differentiability

Proposition

Let $F : U \subseteq \mathbb{M} \to \mathbb{R}^n$ and $G : U \to \mathbb{R}^n$ be Newton differentiable at $\bar{x}$ with Newton differentials $\mathcal{H}F$ and $\mathcal{H}G$. Then $F + G$ is Newton differentiable at $\bar{x}$ with Newton differential $x \mapsto \{H_F + H_G \mid H_F \in \mathcal{H}F(x), H_G \in \mathcal{H}G(x)\}$

Proposition

Let $F : U \subseteq \mathbb{M} \to \mathbb{R}^n$ and $G : U \to \mathbb{R}^m$ be Newton differentiable at $\bar{x}$ with Newton differentials $\mathcal{H}F$ and $\mathcal{H}G$. Then $F \oplus G : U \to \mathbb{R}^{n+m}$ is Newton differentiable at $\bar{x}$ with Newton differential $x \mapsto \{H_F \oplus H_G \mid H_F \in \mathcal{H}F(x), H_G \in \mathcal{H}G(x)\}$

Calculus of Newton Differentiability

Proposition

Let $F : U \subseteq \mathbb{M} \to \mathbb{R}^m$ and $G : F(U) \to \mathbb{R}^n$ be Newton differentiable at $\bar{x}$ and $F(\bar{x})$ with Newton differentials $\mathcal{H}F$ and $\mathcal{H}G$
Assume F is continuous at $\bar{x}$ and $\exists K > 0$ such that

$$\sup_{x \in U} \sup_{H \in \mathcal{H}F(x)} \sup_{y,z \in U} \|H(x)(y,z)\| \leq K \,\mathrm{d}(y,z)$$

Then $G \circ F$ is Newton differentiable at $\bar{x}$ with Newton differential

$$\mathcal{H}(G \circ F)(x) = \{(y,z) \mapsto H_G(F(y), F(z)) \mid H_G \in \mathcal{H}G(F(x))\}$$

In the context of $F : \mathbb{R}^n \to \mathbb{R}^m$ and
$\mathcal{H}G(x)(y,z) := \nabla G(x)^T (z - y)$

$$\begin{aligned}\|\mathcal{H}(G \circ F)(x)(x, \bar{x})\| &= \|\nabla G(F(x))^T (F(\bar{x}) - F(x))\| \\ &\approx \|\nabla G(F(x))^T \| \nabla F(x)^T \| \| \bar{x} - x\|\end{aligned}$$

Calculus of Newton Differentiability

Proposition

Let $F : U \subseteq \mathbb{M} \to \mathbb{R}$ and $G : U \to \mathbb{R}$ be Newton differentiable at $\bar{x}$ with Newton differentials $\mathcal{H}_F$ and $\mathcal{H}_G$. Then $F \cdot G$ is Newton differentiable at $\bar{x}$ with Newton differential

$$\begin{aligned}\mathcal{H}(F \cdot G)(x) =& \{(y,z) \mapsto H_F(y,z)G(x) + F(x)H_G(y,z) \\ & \mid H_G \in \mathcal{H}G(x), H_F \in \mathcal{H}F(x)\}\end{aligned}$$

Newton-type Methods

Given $F : \mathbb{M} \to \mathbb{R}^n$ Newton differentiable with Newton differential $\mathcal{H}F$ and $\mathcal{H}F(x)$ inversely compatible for all x

Definition (Newton-type Method in Quasi-metric Spaces)

The fixed point iteration of the proper (nowhere empty) set-valued operator $\mathcal{N}_{\mathcal{H}F} : M \rightrightarrows M$,

$$\mathcal{N}_{\mathcal{H}F}x = \{H^{-}(x, -F(x)) \mid H \in \mathcal{H}F(x)\}$$

$$x^{k+1} \in \mathcal{N}_{\mathcal{H}F}x^{k}$$

is called a *Newton-type method*.

In the context of $\mathbb{M} = \mathbb{R}^n$ and $\mathcal{H}F(x) = \nabla F(x)^T$

$$H^{-}(x, -F(x)) = x + \nabla F(x)^{T^{-1}}(-F(x))$$

Convergence Results

Proposition

Let $F : U \subseteq \mathbb{M} \to \mathbb{R}^n$

1. weakly Newton differentiable at $\bar{x}$ with Newton differential $\mathcal{H}F$
2. $F(\bar{x}) = 0$
3. $\sup_{x \in U} |||H^{-}|||c < 1$, where

$$\lim_{x \to \bar{x}} \sup_{H \in \mathcal{H}F(x)} \frac{\|F(x) - F(\bar{x}) - H(x, \bar{x})\|}{\mathrm{d}(x, \bar{x})} = c$$

Then there exists a neighborhood $\bar{x} \in V \subseteq \mathbb{M}$ such that the mapping $\mathcal{N}_{\mathcal{H}F}$ is a quasi-contraction on V

Corollary

$\{x^k\}$ *with* $x^{k+1} \in \mathcal{N}_{\mathcal{H}F}(x^k)$ *converges linearly to* $\bar{x}$

Application: Gradient Descent

Recall: Proposition

$f : \mathbb{R}^n \to \mathbb{R}$ μ-strongly convex $\mathcal{C}^1$ and L-smooth, then ∇f is weakly Newton differentiable at any x with a Newton differential $\mathcal{H}(\nabla f)(x) := \{\alpha\,\mathrm{I} \mid \frac{\sqrt{L^2+\alpha^2-2\alpha m}}{\alpha} < 1\}$

Gradient Descent

The Newton-type method:

$$x^{k+1} = x^k - \alpha \nabla f(x^k)$$

Proposition

When $\sqrt{L^2 + \alpha^2 - 2\alpha m} < \alpha$ then $c|\!|\!| H^- |\!|\!| < 1$

Corollary

$\{x^k\}$ *generated by Gradient Descent converges linearly to* $\bar{x}$

Superlinear Convergence

Proposition

Let $F : U \subseteq \mathbb{M} \to \mathbb{R}^n$

1. Newton differentiable at $\bar{x}$ with Newton differential $\mathcal{H}F$
2. $F(\bar{x}) = 0$
3. $\sup_{x \in U} |||H^{-}||| < \infty$

Then there exists a neighborhood $\bar{x} \in V \subseteq \mathbb{M}$ such that the mapping $\mathcal{N}_{\mathcal{H}F}$ is a quasi-contraction on V and the sequence $\{x^k\}$ with $x^{k+1} \in \mathcal{N}_{\mathcal{H}F}(x^k)$ converges superlinearly

Application: Newton's Method

Recall: Proposition

$f : \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{C}^2(\mathbb{R}^n)$, then ∇f is Newton differentiable at any $x \in \mathbb{R}^n$ with a Newton differential $\mathcal{H}(x) := \{\nabla^2 f(x)\}$

Newton's Method

The Newton-type method:

$$x^{k+1} = x^k - \nabla f(x^k)^{-1} \nabla f(x^k)$$

Proposition

$f : \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{C}^2(\mathbb{R}^n)$, $\nabla f(\bar{x}) = 0$, $\nabla^2 f(\bar{x})$ invertible, then $\{x^k\}$ generated by Newton's method converges super-linearly to $\bar{x}$

Quadratic Convergence

Proposition

Let $F : U \subseteq \mathbb{M} \to \mathbb{R}^n$

1. Newton differentiable at $\bar{x}$ with Newton differential $\mathcal{H}F$ and
$$\lim_{x \to \bar{x}} \sup_{H \in \mathcal{H}F(x)} \frac{\|F(x) - F(\bar{x}) - H(x, \bar{x})\|}{\mathrm{d}(x, \bar{x})^2} < \infty$$
2. $F(\bar{x}) = 0$
3. $\sup_{x \in U} |||H^{-}||| < \infty$

Then there exists a neighborhood $\bar{x} \in V \subseteq \mathbb{M}$ such that the mapping $\mathcal{N}_{\mathcal{H}F}$ is a quasi-contraction on V and the sequence $\{x^k\}$ with $x^{k+1} \in \mathcal{N}_{\mathcal{H}F}(x^k)$ converges quadratically

Application: Newton's Method for Convex Optimization

Proposition

$f : \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{C}^2(\mathbb{R}^n)$, f strongly convex with a global minimum at $\bar{x}$ then ∇f is Newton differentiable at any $x \in \mathbb{R}^n$ with a Newton differential $\mathcal{H}(x) := \{\nabla^2 f(x)\}$ and

$$\lim_{x \to \bar{x}} \frac{\|\nabla f(x) - \nabla f(\bar{x}) - \nabla^2 f(x)(x - \bar{x})\|}{\|x - \bar{x}\|^2} < \infty$$

Newton's Method for Convex Optimization

The Newton-type method:

$$x^{k+1} = x^k - \nabla f(x^k)^{-1} \nabla f(x^k)$$

Proposition

$\{x^k\}$ generated by Newton's method converges quadratically to $\bar{x}$

Application: Quasi-Newton Methods

A quasi-Newton method is characterized by an update rule $\mathcal{U} : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$

Algorithm

$$\begin{cases} x^{k+1} & = x^k - {H^k}^{-1} F(x^k) \\ H^{k+1} & = \mathcal{U}(x^k, H^k) \end{cases}$$

In matrix form:

$$\begin{bmatrix} x^{k+1} \\ H^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ H^k \end{bmatrix} - \begin{bmatrix} {H^k}^{-1} & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} F(x^k) \\ H^k - \mathcal{U}(x^k, H^k) \end{bmatrix}$$

Quasi-Newton Methods

Proposition

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ Newton differentiable at $\bar{x}$ and $\bar{H} \in \mathcal{H}F(\bar{x})$.
Assume

$$\lim_{x \to \bar{x}} \sup_{H_F \in \mathcal{H}F(x)} \|H_F - \bar{H}\| = 0$$

Assume

$$\lim_{x, H \to \bar{x}, \bar{H}} \frac{\|\mathcal{U}(x, H) - \bar{H}\|_F}{\|H - \bar{H}\|_F} = 0$$

Consider $G : U \times \mathbb{R}^{n \times n} \to \mathbb{R}^n \times \mathbb{R}^{n \times n}$

$$G(x, H) = \begin{bmatrix} F(x) & H - \mathcal{U}(x, H) \end{bmatrix}^T .$$

Then G is Newton differentiable at $(\bar{x}, \bar{H})$ with

$$\mathcal{H}G(x, H) = \left\{ \begin{bmatrix} H & 0 \\ 0 & \mathrm{I} \end{bmatrix} \right\}$$

Kantorovich-type Theorem

The Classic Kantorovich Theorem

Theorem

Let $F \in \mathcal{C}^1$ *and define* $B := \|\nabla F(x^0)^{-1}\|$ *and* $\eta := \|\nabla F(x^0)^{-1} F(x^0)\|$.

If

1. F *is L-smooth,*

$$\|\nabla F(x) - \nabla F(y)\| \leq L\|x - y\|$$

2. $2LB\eta \leq 1$

then $\exists \bar{x}$ *near* x^0 *such that* $F(\bar{x}) = 0$

Definitions

We need a stronger notion of inverse compatibility

Definition

An inversely compatible pseudo-linear mapping $H : \mathbb{M} \times \mathbb{M} \to \mathbb{R}^n$ is called strongly inversely compatible if

$$\forall x, \forall v \in \mathbb{R}^n, \quad v = H(x, H^-(x, v))$$

and

$$\forall x, \forall v \in \mathbb{R}^n, \quad x = H^-(x, v) \Rightarrow v = 0$$

The set of all such mappings is denoted by $SGS_n(\mathbb{M})$

Definitions

We also need a weaker notion of L-smooth

Definition

A mapping $H : \mathbb{M} \to GS_n(\mathbb{M})$ is called pointwise h-smooth at x^0 if there exist $\kappa > 0$ and $\alpha > 0$ such that

$$\mathrm{d}(H(x)^{-}(x, H(x^0)(z, y)), x) \leq \left(1 + \kappa \mathrm{d}(x, x^0)^{\alpha}\right) \mathrm{d}(y, z)$$

Remark

When $\mathbb{M} = \mathbb{R}^n$, this is equivalent with Hölder continuity

$$\| T(x)^{-1} T(x^0) \| \leq \left(1 + \kappa' \|x - x^0\|^{\alpha}\right)$$

Kantorovich-type Theorem

Theorem

Let $\mathbb{M}$ complete and F Newton differentiable with single valued Newton differentiable and

$$\|F(x) - F(y) - \mathcal{H}F(x)(y,x)\| \leq \frac{1}{2} L \,\mathrm{d}(y,x)^{\gamma}$$

and define $B := \| \mathcal{H}F(x^0)^{-} \|$ and $\eta := \mathrm{d}(x^0, H_F(x^0)^{-}(x^0, F(x^0)))$

1. *F is h-smooth with $\kappa = L$ and $\alpha = \gamma - 1$,*
2. $$\frac{LB}{2}\left(\frac{f(t)}{f'(t)}\right)^{\gamma} \leq f\left(t - \frac{f(t)}{f'(t)}\right)$$ $$f(0) = \eta, \quad f(t) > 0, \quad f(\bar{t}) = 0,$$ $$f'(t) < 0, \quad f'(t) \geq -(1 + Lt^{\gamma-1})^{-1}, f''(t) > 0$$

then $\exists \bar{x}$ near x^0 such that $F(\bar{x}) = 0$

The Classic Kantorovich Theorem Revisited

In the case $\gamma = 2$, take $f(t) = LBt(t+1) - \eta$
Then

$$\frac{LB}{2}\left(\frac{f(t)}{f'(t)}\right)^{\gamma} \le f\left(t - \frac{f(t)}{f'(t)}\right)$$

$$f(0) = \eta, \quad f(t) > 0, \quad f(\bar{t}) = 0,$$

$$f'(t) < 0, \quad f'(t) \ge -(1 + Lt^{\gamma-1})^{-1}, f''(t) > 0$$

if $2LB\eta \le 1$

Proof.

Step 1 Show that the Newton method applied to f converges superlinearly

Step 2 Define $\Sigma(t) = \{x \mid \|x - x^0\| < t, \|\nabla F(x^0) F(x)\| \leq f(t)\}$

Step 3 Show with recursion $x^k \in \Sigma(t^k)$

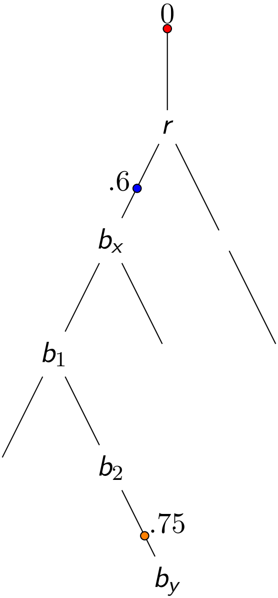
Step 4 Telescope and triangle to get

$$\|x^m - x^n\| \leq t^m - t^n$$

Step 5 Cauchy sequence

□

Example


0
r
.6
b_x
b_1
b_2
.75
b_y

Exmaple

Introduce $f : [0, \infty) \to \mathbb{R}$ strongly convex, $\mathcal{C}^\infty$ with a minimizer in $(0, \max_{(b_x,x)\in\mathbb{M}} \mathrm{d}((r,0),(b_x,x)))$

$$F(b_x, x) = f'(\mathrm{d}((r,0),(b_x,x)))$$

Solving $F(b_{\bar{x}}, \bar{x}) = 0$ finds

$$\operatorname*{argmin}_{(b_x,x)\in\mathbb{M}} f(\mathrm{d}((r,0),(b_x,x)))$$

Define $d_r(b_x, x) = \mathrm{d}((r,0),(b_x,x))$.

Exmaple

Define: $\mathcal{H}F : \mathbb{M} \to S_1(\mathbb{M})$

$$\mathcal{H}F(b_x, x)((b_y, y), (b_z, z)) = f''(d_r(b_x, x))(d_r(b_z, z) - d_r(b_y, y))$$

Then F is Newton differentiable at $(b_{\bar{x}}, \bar{x})$
The next step: check inverse compatibility
Defined

$$\pi(t) = (b_{\lfloor t \rfloor}, t - \lfloor t \rfloor),$$

where $b_0, b_1, \dots, b_{\lfloor d_r(m,1) \rfloor}$
Define

$$\mathcal{H}F^{-}(b_x, x)((b_y, y), v) = \pi(\mathrm{clamp}(d_r(b_y, y) + f''(d_r(b_x, x))^{-1} v)))$$

Example

The Newton-type method

$$(b_{x_{k+1}}, x_{k+1}) = \pi(d_r(b_{x_k}, x_k) - f''(d_r(b_{x_k}, x_k))^{-1} f'(d_r(b_{x_k}, x_k)))$$

Proposition

$\mathcal{H}F$ is Newton differentiable and inversely compatible with $\mathcal{H}F^-$
and $|||\mathcal{H}F^-|||$ is bounded;
$(b_{x_{k+1}}, x_{k+1})$ converges superlinearly to

$$\operatorname*{argmin}_{(b_x, x) \in \mathbb{M}} f(\mathrm{d}((r, 0), (b_x, x)))$$

Conclusions

1. Newton Differentiability (non-smooth analysis and numerical methods)
2. Algebra on (Quasi-)Metric Spaces
3. Newton-type Methods on (Quasi-)Metric Spaces
4. Kantorovich-type Theorem
5. Inverse Function Theorem

How do we use this for optimization?

Thank you!

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