New Optimization Algorithms for Neural Network Training Using Operator Splitting Techniques

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Combining multiple layers we can construct a function from $\mathbb{R}^n$ to $\mathbb{R}^m$

It is proven in[3] that a single layer neural network with suitable weights and biases can approximate any continuous function with an error less than ϵ.

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In order to compute the weights and biases that would yield an interpolation to a function defined on a dataset x^i, y^iwe could minimize a cost function that depends on the weights and biases of our network

$$c(x, y) = \|f(x) - y\|$$

Backpropagation

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A similar technique gives the derivative with respect to the biases

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The convergence rate of the algorithm is $\mathcal{O}(n^{-1})$

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If the expected value of the gradient components is equal to the true gradient we get a convergence rate of $\mathcal{O}(n^{-0.5})$ A thorough exposition of different Stochastic Gradient Descent algorithms is available in[4]

The Heavy Ball with Friction Model

If we consider a dynamic system inspired by a ball with mass that rolls with friction on a surface under the force of gravity we arrive at the ODE

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With a constant γ a finite difference scheme gives Polyak's algorithm

$$y_n = y_{n-1} + h\nabla E(x_{n-1})$$
$$x_n = x_n - \eta y_n$$

For $\gamma = 3t^{-1}$ the discrete dynamic system associated to the ODE yields Nesterov's Accelerated Gradient[6]

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The RMSprop algorithm divides the learning rate with the square root of the cumulated sum of the squares of the gradients.

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The Adadelta algorithm also multiplies the learning rate with the square root of the cumulated sum of the squares of the coordinate updates Δx_n. An analysis of the continuous dynamic systems for adaptive algorithms is presented in[1]

Operator Splitting

Operator Splitting is a technique that approximates the solution of a linear ODE

$$\dot{x} = (A + B)x$$

Normally

$$e^{(At+Bt)} = e^{At}e^{Bt}$$

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A similar bound was proven in[5] for

$$\|e^{(At+Bt)} - e^{Bt/2}e^{At}e^{Bt/2}\| \in \mathcal{O}(t^2)$$

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We use a forward Euler method in order to obtain a discrete version of each problem.

Sequential Splitting Algorithms

Combining the 2 sub-problems, denoting $n/(n+3)$ with β_n and introducing a hyper-parameter k in order to better control the velocity we arrived at the final algorithm SSA1

$$\begin{cases} y_n = u_n + h\beta_n v_n \\ v_{n+1} = \beta_n^k \cdot ((1 - h\beta_n)v_n - h\nabla f(y_n)) \\ u_{n+1} = u_n + \beta_n(1 - h\beta_n)(y_n - u_n) - h^2\nabla f(y_n) \end{cases}$$

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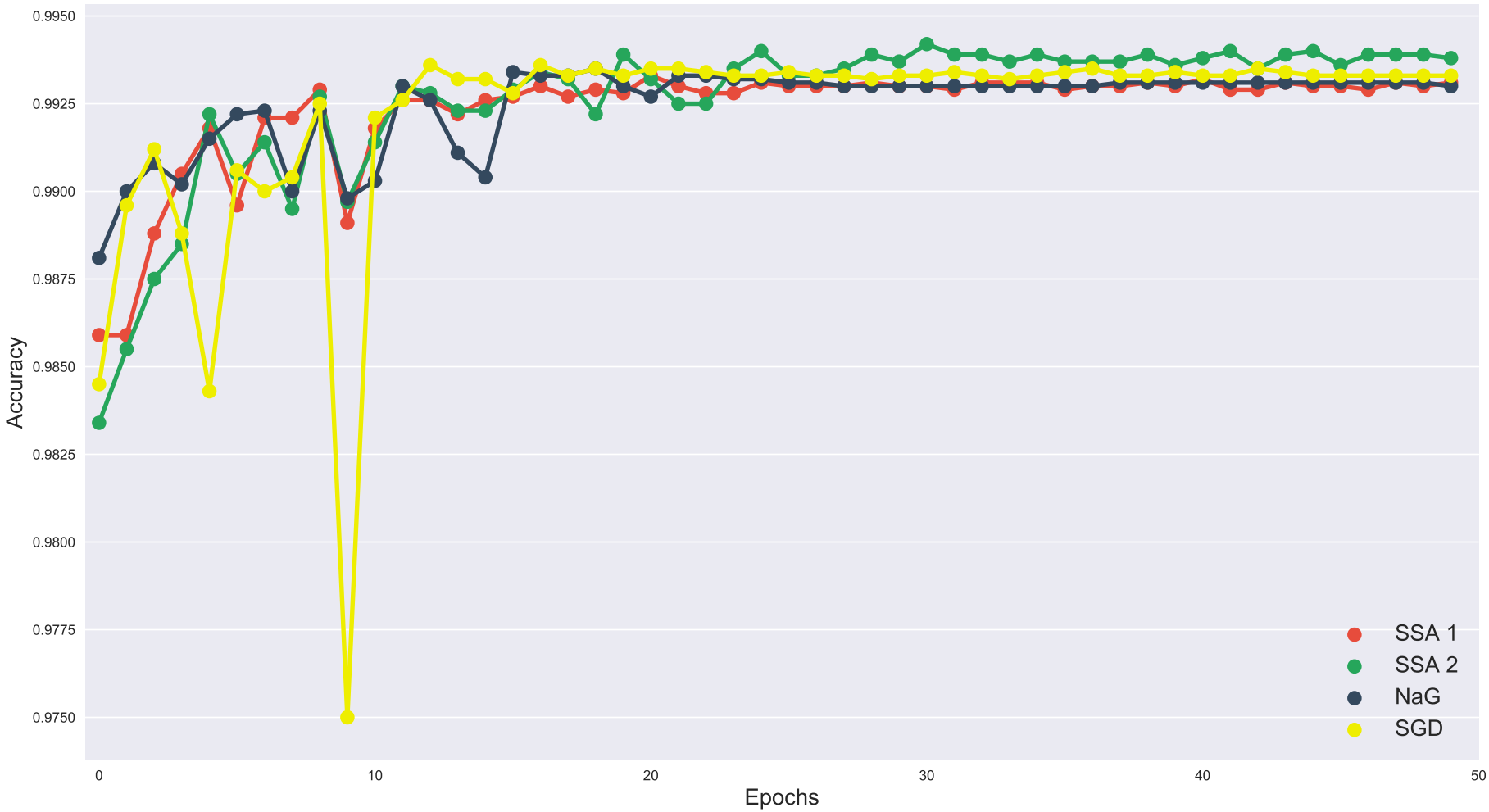
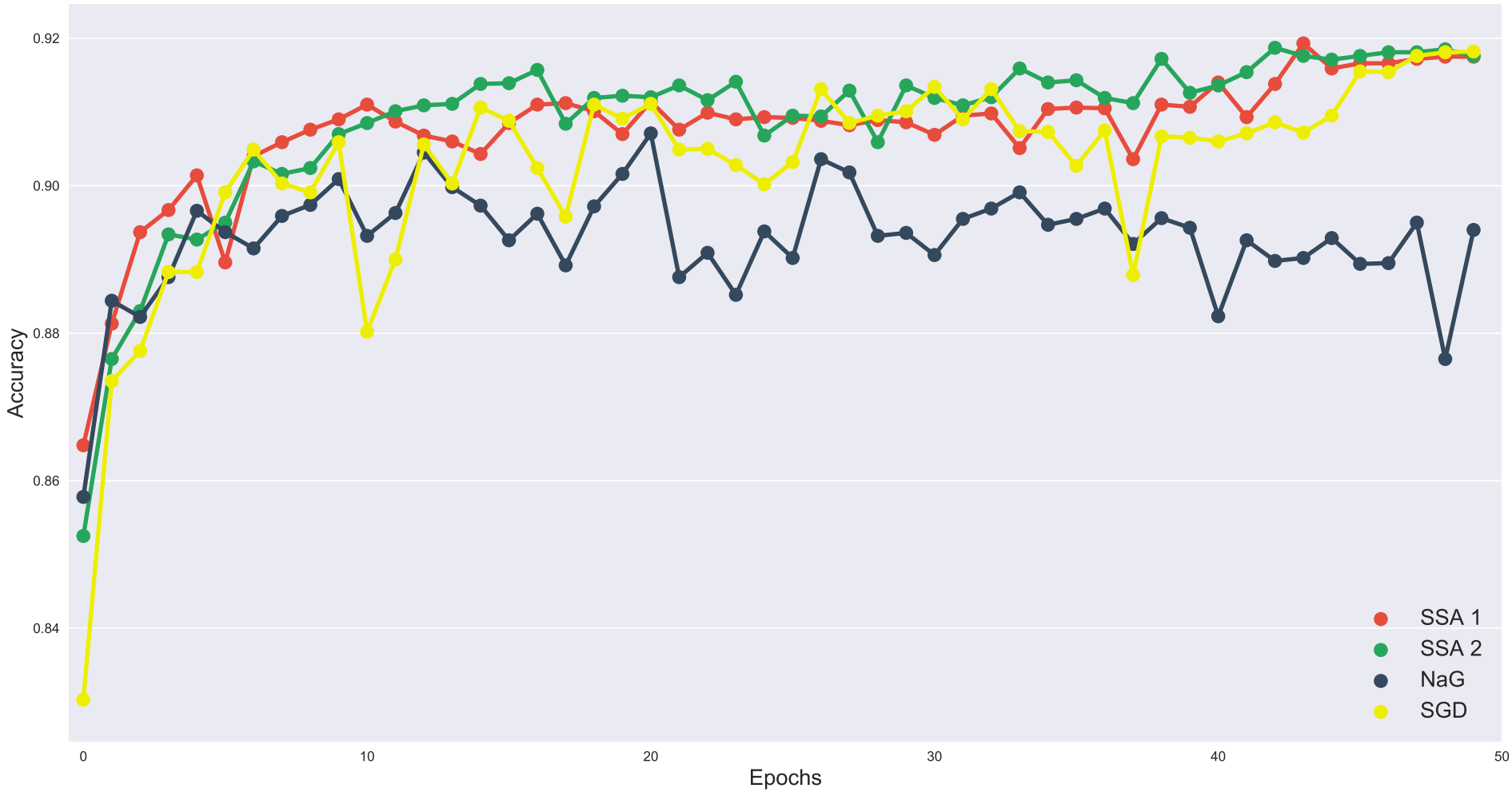
A more detailed construction is available in[2]

Results


0.9950
0.9925
0.9900
0.9875
0.9850
0.9825
0.9800
0.9775
0.9750
Accuracy
0
10
20
30
40
50
Epochs
SSA 1
SSA 2
NaG
SGD

Results


0.92
0.90
0.88
0.86
0.84
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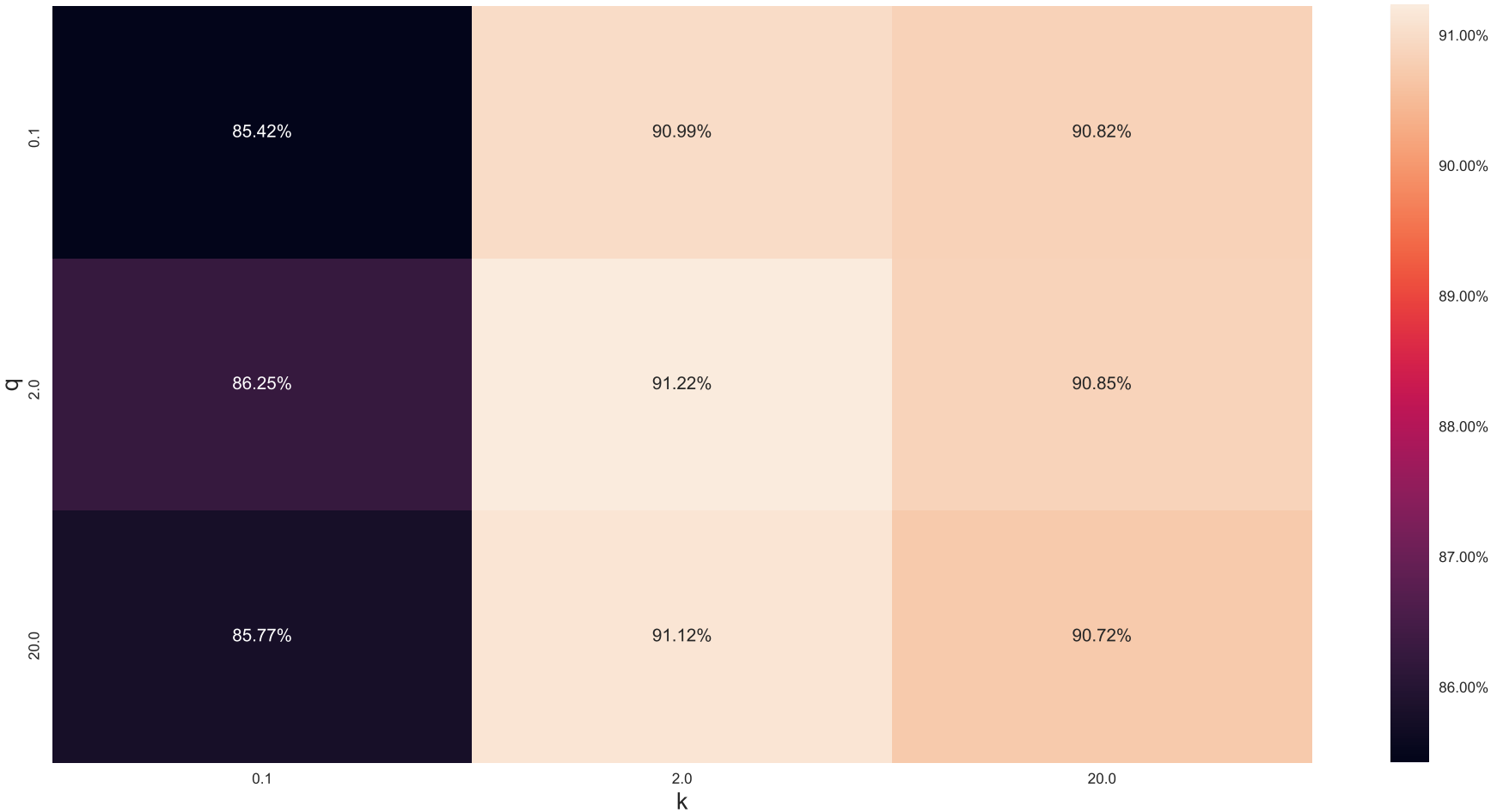
85.42%
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