Non-Euclidean Proximal Algorithms for Quadratic-Composite Optimization: The Case of Mean Curvature Flow

Titus Pinta

Institut für Numerische und Angewandte Mathematik
Universität Göttingen

22 March 2022

DFG Deutsche Forschungsgemeinschaft

siam 2022 Conference on Imaging Science

Outline

1 Mean Curvature Flow

2 Thresholding Scheme for Mean Curvature Flow

3 Proximity Operator Formulation

4 Proximal Splitting Method

5 Bregman Proximal Gradient Formulation

6 Conclusions

Mean Curvature Flow

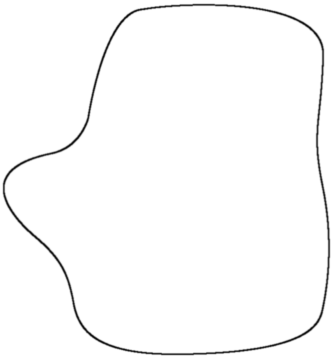

Figure: Mean Curvature Flow

Theoretical Goals

What do we want to do?

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- Model of the Evolution

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- Model of the Evolution
- Numerical Simulation

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- Numerical Simulation
- Compute Limit Points

Setup for Mean Curvature Flow

C^2 immersion of M^n in a N^{n+k} (Riemannian)

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$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial s^2}$$

Properties of the Flow

Similar to the heat equation

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- **Gradient Flow** of the length functional

$$\frac{\partial \gamma}{\partial t} = -\nabla L(\gamma), \quad \text{where } L(\gamma) = \int_\gamma 1 \mathrm{d}s$$

A different Formulation of the Problem

Identify a Jordan curve with $\chi : \mathbb{R}^2 \to [0, 1]$,

$$\chi(x) = \begin{cases} 1 & \text{if } x \text{ is in the inside of a curve} \\ 1 & \text{if } x \text{ is on the curve} \\ 0 & \text{if } x \text{ is outside the curve} \end{cases}$$

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The set of functions is

$$\mathcal{X} = \{\chi : \mathbb{T}^2 \to [0, 1] | \chi \text{ measurable}\}$$

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$$\chi_{n+1} = x \mapsto \begin{cases} 1 & \text{if } u_{n+1}(x) > 1/2 \\ 1 & \text{if } u_{n+1}(x) = 1/2 \\ 0 & \text{if } u_{n+1}(x) < 1/2 \end{cases}$$

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Increment n;

end

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Computation Method for the Thresholds Scheme

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Green's function is $G_h : \mathbb{T}^2 \to \mathbb{R}$

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Thresholding is local

Analysis of One Iteration and Diffusion Generated Motion

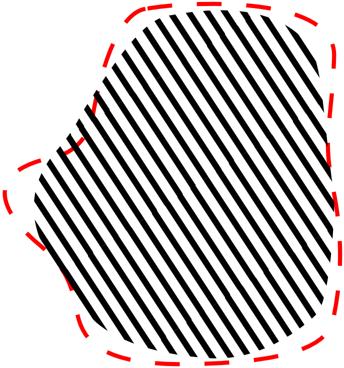

Figure: Diffusion of a Region

Analysis of One Iteration at a Point

Use polar coordinates from the center of the osculating circle

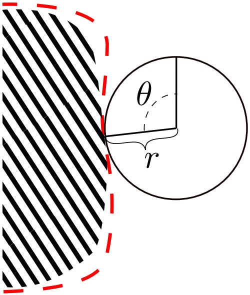

Figure: Polar Coordinates from the Center of the Osculating Circle

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The heat equation is

$$\frac{\partial u}{\partial t} = \frac{1}{R}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{R^2}\frac{\partial^2 u}{\partial \theta^2}$$

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By symmetry, for a point on the level set $u = 1/2$

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Use thresholding to initialize the data for the next iteration

Properties of the Scheme

Order Preservation: $\chi_n \leq \overline{\chi}_n$ then $\chi_{n+1} \leq \overline{\chi}_{n+1}$

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Let $U_h : \mathcal{X} \to \mathcal{X}$,

$$U_h(\chi) = x \mapsto \begin{cases} 1 & \text{if } x \in \{G_h * \chi \geq 1/2\} \\ 0 & \text{if } x \notin \{G_h * \chi \geq 1/2\} \end{cases}$$

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Metric Regularity (Hausdorff metric)

$$\text{dist}(C_1, C_2) \leq \text{dist}(\{U_h(\chi_{C_1}) = 1\}, \{U_h(\chi_{C_2}) = 1\})$$

Convergence

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$\chi_{\gamma(\cdot,t)} \in \mathcal{X}$ the characteristic function of $\gamma(\cdot, t)$
$\gamma(\cdot, t)$ the weak solution of MCF with initial data $\gamma(\cdot, 0)$

Theorem (Evans)

For all $\chi_0 \in \mathcal{X}$, *for all* $t > 0$

$$U^n_{\left(\frac{t}{n}\right)}(\chi_{\gamma(\cdot,0)}) \to \chi_{\gamma(\cdot,t)},$$

uniformly

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What else can we say about this method

- Reformulate using the prox operator
- Prove equivalence
- Show why this reformulation is useful

Proximity Operator Formulation

Let $E_h : \mathcal{X} \to \mathbb{R}$

$$E_h(\chi) = \frac{1}{\sqrt{h}} \int_{\mathbb{T}^2} (1 - \chi) G_h * \chi \mathrm{d}x$$

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and the metric $d_h : \mathcal{X}^2 \to [0, \infty)$

$$d_h^2(\chi, \overline{\chi}) = 2\sqrt{h} \int_{\mathbb{T}^2} (G_{\frac{h}{2}} * (\chi - \overline{\chi}))^2 \mathrm{d}x$$

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$\mathcal{X}$ is compact and E_h is continuous for all $h > 0$

Proximity Operator Formulation

Lemma

$$\arg\min_{\chi \in \mathcal{X}} E_h(\chi) + \frac{1}{2h} d_h(\chi, \chi_n)^2$$

Proximity Operator Formulation

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$$\chi_{n+1} \in \underset{\chi \in \mathcal{X}}{\arg\min}\, E_h(\chi) + \frac{1}{2h} d_h(\chi, \chi_n)^2$$

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$$E_h(\chi) + \frac{1}{2h} d_h(\chi, \chi_n)^2 = \frac{1}{\sqrt{h}} \int_{\mathbb{T}^2} \chi(1 - 2G_h * \chi_n)\mathrm{d}x + \frac{1}{\sqrt{h}} \int_{\mathbb{T}^2} \chi_n G_h * \chi_n \mathrm{d}x$$

□

Γ-convergence of the Energy Functional

Definition (Γ-convergence)

$\Gamma\text{-}\lim F_n = F$ if

$$\begin{cases} \forall x,\ \forall x_n \to x, & F(x) \leq \liminf F_n(x_n) \\ \forall x,\ \exists x_n \to x, & F(x) \geq \limsup F_n(x_n) \end{cases}$$

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$$\Gamma\text{-}\lim E_h(\chi_\gamma) = c \int_{\mathbb{T}^2} \|\nabla \chi_\gamma\| \mathrm{d}x \propto L(\gamma)$$

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Proximal Splitting Method

Can we show convergence using the prox operator formulation?

- Spit in Forwards-Backwards
- Relaxe using Bregman Divergence
- Show global convergence

Proximal Splitting Method

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Step 1:

$$u_{n+1} = G_h * \chi_n$$

Step 2:

$$\chi_{n+1} = x \mapsto \begin{cases} 1 & \text{if } u_{n+1}(x) \geq 0 \\ -1 & \text{if } u_{n+1}(x) < 0 \end{cases}$$

Proximal Splitting Method

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Step 1 is gradient descent for E_D

$$u_{n+1} = \chi_n - h \nabla E_D(\chi_n)$$

Proximal Splitting Method

Let $\bar{\iota}_M : \mathbb{T}^2 \to \overline{\mathbb{R}}$

$$\bar{\iota}_M(c) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{else} \end{cases}$$

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Extend $\iota_M : C(\mathbb{T}^2) \to \overline{\mathbb{R}}$

$$\iota_M(u) = \begin{cases} \infty & \text{if } \exists x \text{ such that } u(x) \notin M \\ 0 & \text{else} \end{cases}$$

Proximal Splitting Method

Step 2 is an application of the prox operator

$$\chi_{n+1} = \mathrm{prox}_{\iota_{\mathbb{S}^0}, h}(u_{n+1})$$

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$$\begin{aligned}\chi_{n+1} &= \mathrm{prox}_{\iota_{\mathbb{S}^0},h}(u_{n+1}) \\ &= \operatorname*{arg\,min}_{u\in L^\infty} \iota_{\mathbb{S}^0}(u) + \frac{1}{2h}\int_{\mathbb{T}^2}(u-u_{n+1})^2 \mathrm{d}x\end{aligned}$$

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A full iteration is a forwards-backwards method

$$\chi_{n+1} = \operatorname{prox}_{\iota_{\mathbb{S}^0},h}(\chi_n - h\nabla E_D(\chi_n))$$

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The Bregman divergence

$$D_B(u,v) = \int_{\mathbb{T}^2} \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}u^2 - \frac{1}{2}\|\nabla v\|^2 - \frac{1}{2}^2 v^2 \\ - (\Delta v - v)(u - v)\mathrm{d}x$$

Bregman Proximal Gradient Formulation

Relax $\iota_{\mathbb{S}^0}$

$$\Phi_\varepsilon(u) = \frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}(u^2-1)^2\mathrm{d}x$$

Expand the relaxed algorithm

$$\chi_{n+1} = \underset{\chi\in\mathcal{X}}{\arg\min}\langle\nabla E(\chi_n),\chi\rangle + \Phi_\varepsilon(\chi) + \frac{1}{2h}\|\chi-\chi_n\|^2$$

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Bregman formulation

$$\chi_{n+1} = \operatorname*{arg\,min}_{\chi\in\mathcal{X}} \langle \nabla E(\chi_n), \chi\rangle + \Phi_\varepsilon(\chi) + \frac{1}{h}D_B(\chi,\chi_n)$$

Convergence Bregman Proximal Gradient

Theorem (Convergence of Bregman Proximal Gradient)

The iteration converges if the Lipschitz-like/Convexity Condition

$$D_E(\chi, \overline{\chi}) \leq L D_B(\chi, \overline{\chi})$$

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For $L = 1$

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For $L = 1$

$$\int_{\mathbb{T}^2} \chi(\chi - \overline{\chi}) \mathrm{d}x \geq 0$$

$\chi, \overline{\chi} \in \{-1, 1\}$ so the integrand is positive always positive.

Outline

1 Mean Curvature Flow

2 Thresholding Scheme for Mean Curvature Flow

3 Proximity Operator Formulation

4 Proximal Splitting Method

5 Bregman Proximal Gradient Formulation

6 Conclusions

Recap

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1. A geometric flow given by a parabolic PDE
2. A numerical method for the MCF
3. A proximal splitting of this method method
4. A global convergence analysis,using the Bregman Divergence

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